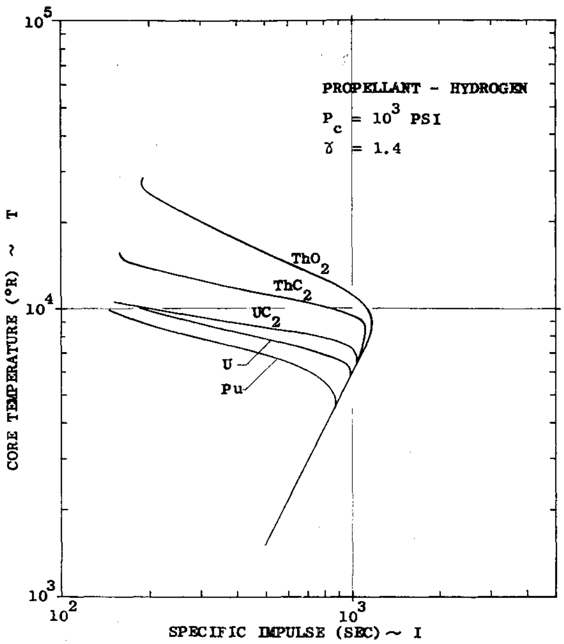

Fig. 3 Specific impulse vs core temperature for several materials.

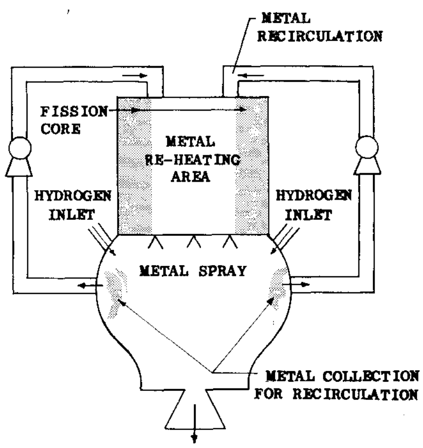

Fig. 4 Liquid-core nuclear rocket: spray concept.

are of this order; therefore, this restriction will probably be only of academic interest.

It has been assumed in these calculations that the partial pressure of fission fuel in the rocket pressure chamber is equal to the liquid's vapor pressure. This may not be the case if the liquid surface is incapable of producing vapor at the rate fission fuel is exhausted through the nozzle.

Kinetic theory yields an expression for the rate matter leaves the liquid surface[3]:

$$\frac{dW_a}{dt} = A(P_f - P_a)\left[\frac{M_f g}{2\pi RT}\right]^{1/2}$$

where A is the fission fuel surface area plus an area term due to the bubble surfaces, and P_a is the actual fuel partial pressure.

The rate mass is exhausted from the nozzle:

$$\frac{dW_f}{dt} = \frac{\text{total fission fuel exhausted through nozzle}}{\text{rocket operating time}}$$

This may be expressed as

$$\frac{F}{I}\frac{P_a M_f}{P_c M_H + P_a(M_f - M_H)}$$

The ratio of dW_a/dt to dW_f/dt is

$$\frac{AP_c}{F}\left[\frac{\gamma\langle M\rangle}{(\gamma - 1)\pi M_f}\right]^{1/2}\left[\frac{P_f}{P_a} - 1\right]$$

At equilibrium vapor pressure, this ratio equals one. An estimate of P_f/P_a may be made by assuming that, if $M_f \sim 10^2$, $\langle M\rangle \sim 2$, $A \sim 10$ ft^2, $P_c \sim 10^3$ psi, $F \sim 10^5$ lbf, then $P_a = \frac{1}{2}P_f$. Since the maximum specific impulse occurs when P_f/P_c is about 10^{-3}, the use of P_f vice P_a will make a negligible difference in the specific impulse maximum.

Several materials and engine designs have been considered, and it appears that the maximum specific impulse of the liquid-core nuclear rocket will be in the range of 1200 to 1400 sec.

References

[1] Rom, F. E., "Advanced reactor concepts for nuclear propulsion," Astronautics 4, 20–22 (October 1959).

[2] Glasstone, S., *Nuclear Reactor Engineering* (D. Van Nostrand Co. Inc., Princeton, N. J., 1955), 1st ed., p. 39.

[3] Kennard, E. H., *Kinetic Theory of Gases* (McGraw-Hill Book Co. Inc., New York, 1958), p. 63.

[4] Vaughan, D. A., "Reactor Core Materials," Battelle Memorial Inst., 4, no. 4, 13 (1961).

[5] Rauh, E. G. and Thorn, R. J., "Vapor pressure of uranium," J. Chem. Phys. 22, 1414–1420 (1954).

[6] Katz, J. J. and Rabinowitch, E., *Chemistry of Uranium* (Dover Publications Inc., New York, 1951), 1st ed., p. 262.

[7] Finnston, H. M. and Howe, J. P., "Metallurgy and fuels," *Nuclear Energy Series* (Pergamon Press, New York, 1956), Vol. 1, no. 5, p. 384.

A Uniqueness Theorem for the Nonlinear Axisymmetric Bending of Circular Plates

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I. Introduction

In this note, the nonlinear axisymmetric bending of circular plates is considered within the scope of the von Kármán theory.[1] It is assumed that the plate is deformed by a symmetrically distributed pressure applied normal to one face. For a variety of boundary conditions along the edge of the plate, the von Kármán equations reduce to a coupled pair of second-order nonlinear ordinary differential equations. The uniqueness theorem given in Ref. 2 for the solution of one of these boundary value problems is valid only for a limited range of parameters. Morosov's[3] uniqueness theorem is established with the aid of the Hildebrand-Graves theorem.

In this paper, an "elementary" proof is given of the uniqueness of the symmetric solutions of von Kármán's equations. For simplicity, only one set of boundary conditions is considered. However, with suitable modifications, uniqueness for other boundary conditions can also be proved. First, the boundary value problem is cast in a form similar to that used by Friedrichs and Stoker[4] for the buckling of circular plates. The uniqueness proof then follows directly from the form of the potential energy functional.

Received July 30, 1963. This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the U. S. Army Research Office (Durham) Grant No. DA-ARO-(D)-31-124-G344. Reproduction in whole or in part is permitted for purposes of the United States Government.

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2. Formulation

Consider a circular plate of thickness t and radius R, which is subjected to a lateral pressure p that varies only with the radial coordinate r. The assumption of axisymmetric deformations implies that the radial and circumferential membrane stresses σ_r and σ_θ and the radial and lateral displacements u and w are functions of r only. The dimensionless variables x, $y(x)$, and $z(x)$ are defined through the relations

$$\begin{aligned} x &\equiv r/R \\ y(x) &\equiv -[12(1-\nu^2)]^{1/2}(R^2/t)(1/r)[dw(r)/dr] \\ [-12(1-\nu^2)/E](R/t)^2\sigma_r(r) &= z(x) \\ [-12(1-\nu^2)/E](R/t)^2\sigma_\theta(r) &= (d/dx)xz(x) \end{aligned} \tag{1}$$

where ν and E are Poisson's ratio and Young's modulus. In terms of these variables,† the von Kármán equations are

$$Gy(x) = -y(x)z(x) - P(x) \tag{2a}$$

$$Gz(x) = \tfrac{1}{2}y^2(x) \tag{2b}$$

Here G is the linear differential operator:

$$G \equiv (1/x^3)(d/dx)x^3(d/dx)$$

and the "loading" function $P(x)$ is given by

$$P(x) = \frac{[12(1-\nu^2)]^{3/2}}{E}\left(\frac{R}{t}\right)^4 \frac{1}{x^2}\int_0^x \xi p(\xi)d\xi$$

To complete the formulation, conditions are required at the center, $x = 0$, and the edge, $x = 1$. Symmetry and regularity of the solutions at the origin imply that

$$y_x(0) = z_x(0) = 0 \tag{3}$$

where a subscript indicates differentiation. At the edge, a variety of conditions may be imposed. For simplicity of presentation, only the following condition is considered:

$$y(1) = z(1) = 0 \tag{4}$$

This implies, from Eq. (1), that the edge is clamped and the radial stress there vanishes. With slight modifications analysis applies to other edge conditions, e.g., clamped with zero radial displacement, simply supported with zero radial stress, and simply supported with zero radial displacement.

Equation (2b) is integrated and Eq. (3) is used to obtain

$$z_x(x) = \frac{1}{2x^3}\int_0^x y^2(\xi)\xi^3 d\xi \tag{5}$$

Then $z(x)$ is given, with the aid of (5), as a functional of $y(x)$ by the relation

$$z(x) = -\int_x^1 z_\xi(\xi)d\xi \tag{6}$$

The *boundary value problem B* is formulated as follows: find a function $y(x)$ that possesses a continuous second derivative and satisfies the differential equation (2*a*) and the boundary conditions

$$y_x(0) = y(1) = 0 \tag{7}$$

The function $z(x)$ in Eq. (2a) is defined by Eqs. (5) and (6). These equations imply that $z(x)$ satisfies Eq. (2b), $z_x(0) = 0$, and $z(1) = 0$. Hence, the boundary value problem B is equivalent to that defined by Eqs. (2–4). The formulation presented here is analogous to the one introduced by Friedrichs and Stoker in their study of the buckling of circular plates[4]; see also Wagner.[5]

† The independent variables $\alpha(x)$ and $\gamma(x)$ in Ref. 2 are given by $\alpha(x) = xy(x)$ and $\gamma(x) = xz(x)$.

The total potential energy can be written as a constant multiplying the functional

$$V[y(x)] = \int_0^1 [y_x^2(x) + z_x^2(x) - 2P(x)y(x)]x^3 dx \tag{8}$$

where $z_x(x)$ is defined in Eq. (5). Wagner[5] has shown, under suitable admissibility conditions on $y(x)$, that V has a minimum, and that the function that minimizes V possesses a continuous second derivative and solves B.

3. Uniqueness Theorem

Let $y^0(x)$ be any solution of B. Let $y(x)$ be any *other* function with a continuous second derivative that satisfies the same boundary conditions as $y^0(x)$. If $y^*(x)$ is defined by the relation

$$y(x) = y^0(x) + y^*(x)$$

it then follows from Eq. (8) and the condition that y^0 is a solution of B, that

$$V[y] - V[y^0] = \int_0^1 [y_x^{*2} + 2z_x^0 z_x^* + (z_x^* + z_x')^2]x^3 dx \tag{9a}$$

Here,

$$\left.\begin{aligned} z_x^0(x) &= \frac{1}{2x^3}\int_0^x y^{02}(\xi)\xi^3 d\xi \geq 0 \\ z_x'(x) &= \frac{1}{x^3}\int_0^x y^0(\xi)y^*(\xi)\xi^3 d\xi \\ z_x^*(x) &= \frac{1}{2x^3}\int_0^x y^{*2}(\xi)\xi^3 d\xi \geq 0 \end{aligned}\right\} \tag{9b}$$

It then follows from Eq. (9) and the positiveness of z_x^0 and z_x^* that,

$$V[y] - V[y^0] \geq 0 \tag{10}$$

Here, the equality holds if and only if $y^*(x) \equiv 0$, i.e., $y(x) \equiv y^0(x)$.

It is now possible to state our uniqueness theorem: *there is, at most, one function that solves B*. To prove this result, it is assumed that there are two solutions, $y^0(x)$ and $y^1(x)$, of B. It then follows from Eq. (10) that

$$V[y^1(x)] \geq V[y^0(x)]$$

and

$$V[y^0(x)] \geq V[y^1(x)]$$

This implies that $V[y^1(x)] = V[y^0(x)]$ and hence, $y^0(x) \equiv y^1(x)$, and the theorem is thus proved.

4. Remarks

The technique of proof presented here may also be applicable to other nonlinear boundary value problems, e.g., we have obtained a related uniqueness theorem for the symmetric deflections of circular membranes using the nonlinear Föppl theory. If unsymmetric solutions are permitted, then according to Ref. (6) there may be boundary conditions for which the von Kármán equations do not have a unique solution.

References

1 von Kármán, T., "Festigkeitsprobleme im Maschinenbau," *Encyklopädie der Mathematischen Wissenschaften* (B. G. Teubner, Leipzig, 1910), Vol. IV, No. 4, pp. 348–352.

2 Keller, H. B. and Reiss, E. L., "Iterative solutions for the non-linear bending of circular plates," Communs. Pure and Appl. Math. 11, 273–292 (1958).

3 Morosov, N. F., "The uniqueness of the symmetrical solution in the problem of the large deflection of a symmetrically loaded circular plate," Soviet Phys. Doklady, **3,** 1275–1278 (1959), transl. from AN USSR **123,** 417–419 (1958).

[4] Friedrichs, K. O. and Stoker, J. J., "The non-linear boundary value problem of the buckled plate," Amer. J. Math. **63**, 839–888 (1941).

[5] Wagner, N. R., "Existence proof for a nonlinear boundary value problem in ordinary differential equations," Ph.D. dissertation Mass. Inst. Tech. (1961).

[6] Yanowitch, M., "Non-linear buckling of circular elastic plates," Communs. Pure and Appl. Math. **9**, 661–672 (1956).

Low-Frequency Acoustic Oscillatory Combustion

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A description is made of a large, double-end burner that is used for the study of low-frequency combustion instability. Also described are the techniques applied and some results obtained by the use of the system.

Introduction

OSCILLATORY combustion is a problem that has frequently been encountered in the development of solid propellant rockets. Accordingly, techniques were developed (for example, Refs. 1–4) whereby oscillatory combustion was studied on a laboratory scale in self-excited systems. Subsequently, techniques have been developed whereby externally excited systems are used to study the more stable propellants.[5,6]

In general, the foregoing techniques have been suitable for studying either the higher acoustic (above 500 cps) frequencies or nonacoustic frequencies. Because of advances in the state of the art, larger motors are now being built. Although a great deal has been learned about acoustic oscillatory combustion in the smaller motors (which exhibit higher frequencies), there is virtually nothing known about acoustic oscillatory combustion at the low frequencies that are of concern in large motors. There has been no laboratory test that would reveal the information, nor has it been gained from motor firings, as they have not yet been made.

Knowledge gained through the trial-and-error methods of development programs is not only extremely costly and time consuming, but it is also generally inadequate for research purposes, and, accordingly, new techniques have been developed.[7,8] With these techniques, it is possible to study acoustic oscillatory combustion in the 5- to 120-cps range with all propellants. This paper is devoted to a discussion of the test burner, techniques, and results obtained.

Apparatus

The burner employed (see Fig. 1) is a $5\frac{1}{2}$-in.-i.d. cylindrical steel tube that is segmented so that it may be used in lengths of 12, 24, 36, 48, or 60 ft. Propellant grains $5\frac{3}{8}$-in. o.d. and usually 1-in. thick (about 2 lb) are used in the burner. The grain is cemented into the end of the burner with epoxy resin. In a given test a single grain may be used in one end of the burner, or grains may be placed in both ends of the burner. Ignition is accomplished by means of an electrically heated bridge-wire that ignites a pyrotechnic paste spread on the propellant surface and coated on the wire.

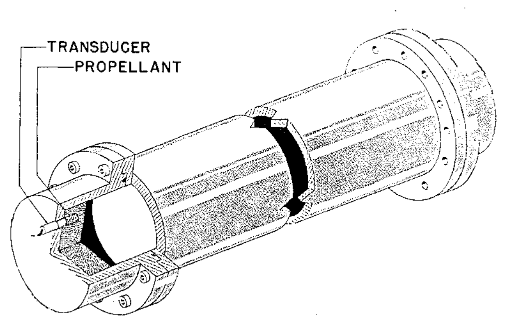

Fig. 1 Test burner used to evaluate a propellant for low-frequency, oscillatory combustion.

The rather simple instrumentation consists of two channels. The main channel is a low-frequency-response pressure transducer whose output signal is recorded on a galvanometer oscillograph. Also recorded on the oscillograph is the amplified and filtered output signal from a high-frequency-response pressure transducer. This latter channel largely performs a "back-up" function for the former channel. Information concerning the pressure-time history (including the pressure oscillations) is taken from the oscillograph record.

Techniques

Depending upon the information desired from the test and the nature of the propellant, there are three types of experimental techniques that are employed. These involve tests that probe the pressure-frequency spectrum for areas of self-excited instability, tests that yield the types of quantitative data which were previously obtained only at higher frequencies,[9] and tests of propellants that are stable in the system so that the propellants may be rated as to their degree of stability.

The exploratory type of test is performed by firing the test in an unvented burner. Following ignition, the closed system increases in pressure until the propellant is consumed. The test will show if, over the pressure range tested and at the characteristic frequency of the burner, the propellant combustion will generate acoustic pressure oscillations in the system. When it is desired that the rate of pressurization in the burner be small, a small bleed or "pseudo-nozzle" is placed midway between the burner ends. Figure 2 shows a record from a test of this nature. By the use of the various burner lengths, the low-frequency end of the spectrum can be probed systematically.

Using the burner, one can also extend to lower frequencies the data that characterize the degree of instability of the propellant. In order to obtain these data, it is necessary that the system be brought to pressure rapidly after ignition and that the oscillations be allowed to grow spontaneously. Because of large free volume and high heat transfer, the sys-

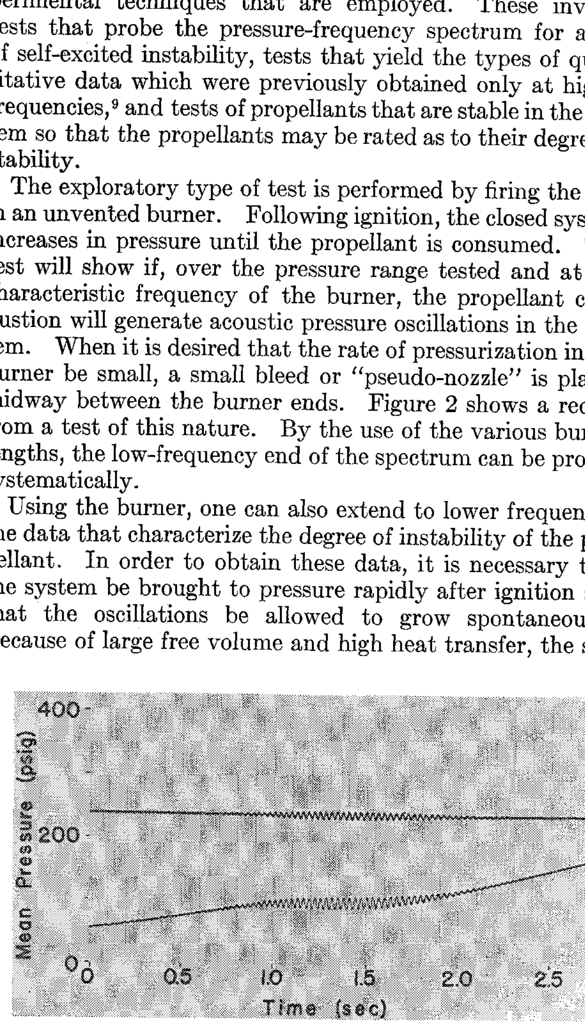

Fig. 2 Portion of a test record obtained from a firing in the sealed burner. The propellant employed showed a marked selectivity in the conditions under which it would oscillate.

Received August 5, 1963. The work was performed under NASA Contract W-11,294-B and Special Projects Task Assignment 71402.

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